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Survey of the eigenfunctions of a billiard system between integrability and chaos

Tomaž Prosen and Marko Robnik

Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SLO-62000 Maribor, Slovenia

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Abstract. We study numerically the eigenfunctions and their Wigner phase space distributions of the two-dimensional billiard system defined by the quadratic conformal image of the unit disk as introduced by Robnik (1983). This system is a *generic* KAM system and displays a transition from integrability to almost ergodicity as the billiard shape changes. We clearly identify two classes of states: the regular ones associated with integrable regions and the irregular states supported on classically chaotic regions, whilst the mixed type states were not found, in support of Percival's conjecture (1973). We confirm the existence of (extremely) intense scars in the classically chaotic regions, and demonstrate their association with classical periodic orbits. Three classes of scars are revealed: one-orbit scars, many-orbit-one-family scars (of statistically similar orbits in the homoclinic neighbourhood), and many-orbit-many-family scars. We argue that it is impossible to find an *a priori* semiclassical theory of *individual eigenstates*, but do not deny the usefulness of general semiclassical arguments in analysing the collective and statistical properties of eigenstates.

1. Introduction

The bound-state eigenfunctions of classically ergodic Hamiltonian systems possess a much less organized structure than those of classically integrable systems. In fact, they have been predicted to be random (Berry 1977, Shnirelman 1979, Voros 1979), such that according to Berry (1977) and Voros (1979) the corresponding mean probability density is determined by the microcanonical Wigner (phase space) function in the semiclassical limit when $\hbar \to 0$. In almost every point in the configuration space of an ergodic system there are an infinite number of possible velocity directions for classical trajectories passing through that point. The correspondence principle suggests then that in the semiclassical limit the eigenfunction can be *locally* represented as a superposition of infinitely many plane waves propagating along the rays—the classical trajectories. *If* their phases are random then one has a random superposition of infinitely many plane waves, giving rise to a Gaussian random function by the central limit theorem. However, Berry's random phase hypothesis (Berry 1977) breaks down on dynamical grounds, possibly as a consequence of the quantum integrability, as has recently been argued by Robnik (1986, 1988, 1989) in analysing the consequences of quantum integrability. It has been demonstrated, using arguments from the perturbation theory, that (Robnik 1986) 'almost every quantum Hamilton system with purely discrete spectrum is (quantum) integrable, but its quantum integrals of motion (specifically, the Weyl correspondents of the operators representing the constants of the motion) generically do not have a classical limit when $\hbar \to 0$'. The qualitative argument (Robnik 1988) is

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that the existence of quantum integrals of the motion at finite non-zero $\hbar$ would imply localization properties of the Wigner distribution (for the eigenstates) within 'quantum tori' near classically periodic orbits, implying the existence of scars. In other words, the existence of quantum integrals of motion at finite $\hbar$ implies subtle correlations between the phases on dynamical grounds, giving rise to deviations from the Gaussian randomness of the eigenfunctions of the classically ergodic (but nevertheless quantum integrable) systems (Robnik 1988).

The early numerical evidence by McDonald and Kaufman (1979) seemed to agree with the property of Gaussian randomness, though some substantial extra probability density has been noticed by McDonald in his unpublished PhD thesis. The importance of the phenomenon of scars as observed in the numerically calculated eigenfunctions of the stadium has been recognized for the first time, however, by Heller (1984), who offered a theory of scars in terms of the wave-packet dynamics (coherent states). He proved the existence of scars in the individual eigenstates by using a statistical argument. It became clear (Heller 1986, Heller *et al* 1987) that scars exist in the eigenfunctions of arbitrarily high-lying states, although the counting measure of the scarred states might vanish with increasing energy (O'Connor and Heller 1988) in consistency with the results of Shnirelman (1979): the quantum expectation value of a smooth operator with classically ergodic dynamics is given by the classical microcanonical average for almost all eigenstates. For a review see Heller (1991).

A semiclassical theory of scars has been elaborated by Bogomolny (1988) in configuration space, and by Berry (1989) in the phase space. Further developments have been published by Aurich and Steiner (1991,1993) for the point particle sliding freely on a two-dimensional compact surface of constant negative curvature. A simple theory has been presented also in (Robnik 1989).

The properties of wavefunctions in the intermediate region between integrability and chaos have been little studied. Our aim in this work is also to survey this transition region in the two-dimensional billiard system† defined by the quadratic conformal image $w(z) = z + \lambda z^2$ of the unit disk $|z| \leqslant 1$ as introduced in (Robnik 1983, 1984) and further explored by Berry and Robnik (1986), Robnik (1992a,b) and Prosen and Robnik (1993a). We choose units such that $\hbar^2/2m = 1$, where m is the mass of the billiard point particle, so that we solve the eigenvalue problem $\Delta\psi + E\psi = 0$ with Dirichlet boundary conditions. We verify numerically the appropriateness of classifying the states exclusively in either regular or irregular states in the transition region. We study the scars and their relation to the classical periodic orbits in chaotic regions, especially in the fully chaotic regime (almost ergodic), and the localization properties of the eigenstates in regular regions.

2. Results

Our primary goal was to investigate systematically the abundance and the intensity of scars in the high-lying eigenstates in the classically fully chaotic regime such as observed in our billiard for $\lambda = 0.375$, which is almost ergodic, in the sense that the tiny stability islands discovered by Hayli *et al* (1987) are not resolved numerically. We have searched for scars in 20 consecutive states (from the 2000th to the 2019th even-parity state), and to our

† Since the billiard boundary is analytic this system is a truly *generic* system (in contrast to non-smooth billiards like the stadium of Bunimovich or the Sinai billiard, or other fully chaotic non-generic systems), where the KAM theory is applicable, and it is a good candidate for investigating the so-called *coexistence problem* of nonlinear dynamics (Strelcyn 1991), namely to show that the irregular regions (of positive Lyapunov exponents) and the regular regions (of vanishing Lyapunov exponents) have positive measure.

slight surprise we found only one brilliant example of an intense scar shown in figure 1(*c*). However, most of the states are nearly Gaussian random like the chaotic state shown in figure 1(*a*), for which the Wigner distribution in an appropriate phase space is expected to be almost microcanonical. We realized that in our billiard the most convenient choice for a classical surface of section used to define the appropriate quantum (Wigner) phase space is represented by the abscissa having the role of the line of section.

The Wigner function (of an eigenstate $\psi(x, y)$) defined in the full phase space (x, y, p_x, p_y) is

$$\rho(x, y, p_x, p_y) = \frac{1}{(2\pi\hbar)^2} \int \mathrm{d}^2 v \, \exp\left(\frac{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{v}}{\hbar}\right) \psi^*\left(x + \frac{v_x}{2}, y + \frac{v_y}{2}\right) \psi\left(x - \frac{v_x}{2}, y - \frac{v_y}{2}\right). \tag{1}$$

In order to compare the quantum Wigner functions with the classical Poincaré maps on the SOS we define the following projection of (1)

$$\rho_{\mathrm{SOS}}(x, p_x) = \int \mathrm{d}p_y \, \rho(x, 0, p_x, p_y) \tag{2}$$

which nicely reduces the number of integrations by one and is equal to

$$\rho_{\mathrm{SOS}}(x, p_x) = \frac{1}{2\pi\hbar} \int \mathrm{d}v_x \, \exp\left(\frac{\mathrm{i}p_x v_x}{\hbar}\right) \psi^*\left(x + \frac{v_x}{2}, 0\right) \psi\left(x - \frac{v_x}{2}, 0\right). \tag{3}$$

As is well known this function is not positive definite and indeed we typically find quite wild oscillations around zero whose average actually vanishes. In a straightforward plot of this object, this fact implies a lot of irrelevant structure which obscures the physical content. Therefore we used the technique of Gaussian smoothing of the Wigner functions ρ_{SOS}, by choosing an appropriate sigma such that the irrelevant oscillations are suppressed while the important structure is preserved. (Note that this is a Husimi-type representation (Takahashi 1989, Heller 1991), but the effective area of our Gaussian is notably smaller than $2\pi\hbar$.) In figures 1(*b*) and 1(*d*) we show (smoothed) ρ_{SOS} for the corresponding eigenstates shown in figures 1(*a*) and 1(*c*). Since the classical SOS is completely chaotic (almost ergodic and therefore not shown) we expect uniform (microcanonical) ρ_{SOS}. This trend is indeed observed in figure 1(*b*) but one should notice the filamentary structure of (smoothed) ρ_{SOS} which we find is quite typical for chaotic states at *intermediate scales*. On the other hand figure 1(*d*) is a brilliant example of a scar in the phase space associated with the shortest and unstable periodic orbit.

We have studied in general the association of the localization regions with the classical periodic orbits in SOS and have plotted all the shortest periodic orbits up to a given period $n_{\max}$ (as defined by the number of bounces off the boundary), which satisfy what we call the *coincidence criterion*. This criterion demands that smoothed ρ_{SOS} at every point of a periodic orbit is larger than 15% of its maximum value in the given eigenstate. In figure 1(*b*) we see that many orbits qualify according to the coincidence criterion while the scar in figure 1(*d*) is supported only by the period-two orbit clearly identified in figure 1(*c*). As will be seen later on this is rather atypical as the scars are in general supported by many orbits. The numerical inspection of this period-two orbit shows that the (smoothed) Wigner function *is precisely localized on the stable and unstable manifolds* which becomes clearer in figure 1(*e*), where the sigma of the smoothing Gaussian is reduced by a factor 2.5. This observation again confirms the findings of Waterland *et al* (1988).

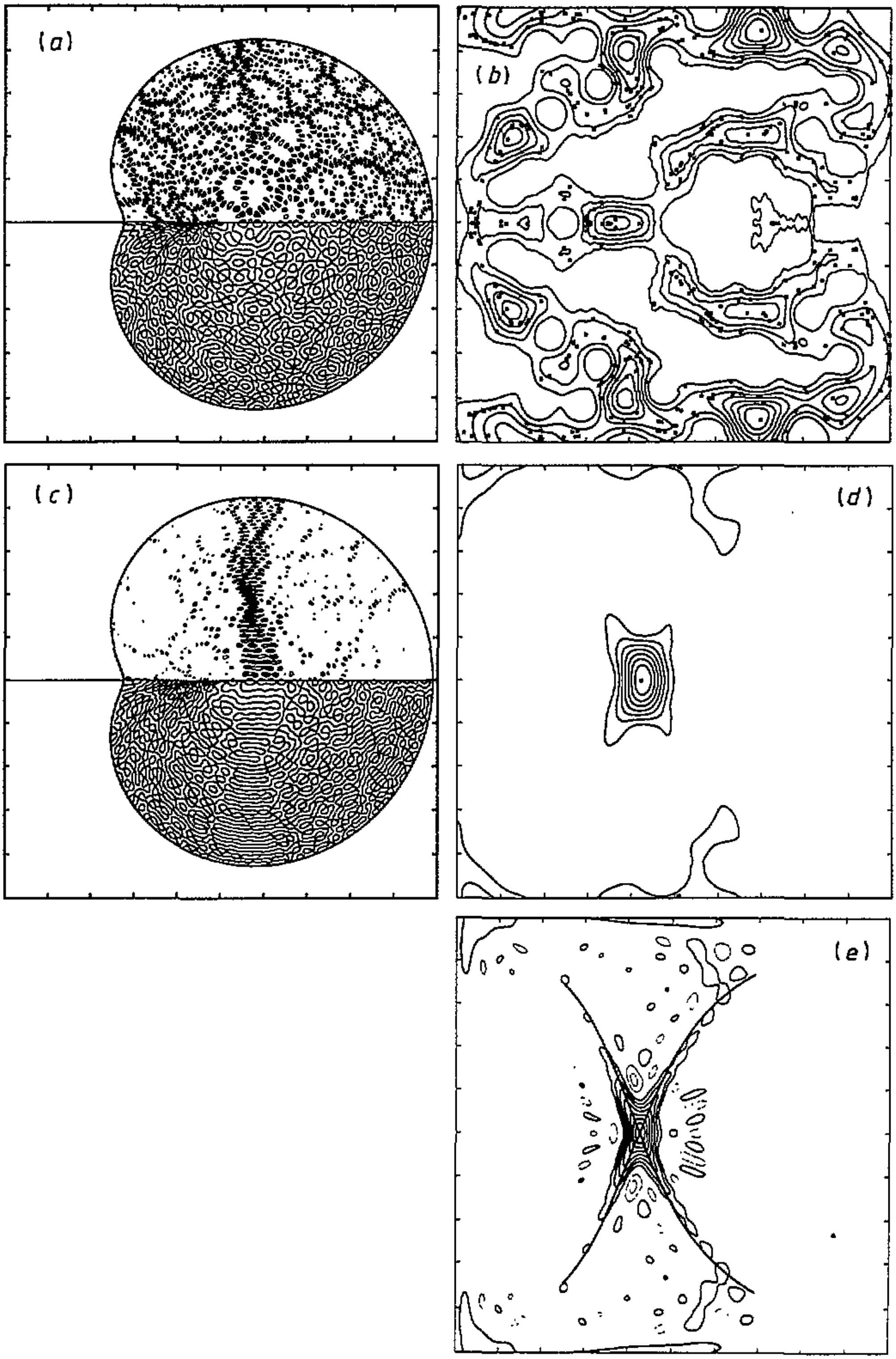
An example of the transition region between integrability and chaos is $\lambda = 0.175$ shown in figure 2, where the classical SOS is plotted in figure 2(*e*). We have examined twenty consecutive states between the 2800th and 2819th even-parity states. Two classes of states are considered. In figure 2(a) we show a regular state associated with the islands of stability in the vicinity of the stable period-four orbit. According to the coincidence criterion several similar orbits support the localization regions. The periods of the longer orbits are multiples of the fundamental period four and they live in its neighbourhood. The second class of states consists of the irregular states which are embedded in the classically chaotic regions, where they can be uniform (ergodic-like) or also exhibit localization regions (scars). One example of a localized irregular state is shown in figures 2(c) and 2(d). The scar clearly observed in figure 2(d) is supported by an unstable period-five orbit and by a family of similar but longer orbits in its homoclinic neighbourhood. This family of orbits—which has been checked numerically to belong to the homoclinic neighbourhood of the primary orbit—very closely resembles the structure of the smoothed Wigner function. We think that this is a clear example demonstrating that in general it is insufficient to describe a scar by only one periodic orbit. As has been suggested already by Robnik (1989) we distinguish three classes of scars:

(i) *One-orbit scars*, i.e. those supported by one classical periodic orbit only (see figures 1(c) and 1(d)).
(ii) *Many-orbit-one-family scars*, i.e. those supported by a dominant unstable periodic orbit and its daughter orbits in its homoclinic neighbourhood (see figures 2(c) and 2(d)).
(iii) *Many-orbit-many-family scars*, i.e. those supported by many different families of statistically similar orbits (see figure 3).

We offer yet another example of intermediate KAM regime of a slightly chaotic billiard for $\lambda = 0.1$. Here we have surveyed twenty consecutive even-parity states (from the 2000th to the 2019th), and we find again that eigenstates belong either to the regular class or to the irregular class. An example of the latter is presented in figures 4(a) and 4(b) where we see excellent overlap of the smoothed Wigner function with the classically chaotic region, and also with the classical periodic orbits selected by the coincidence criterion. One should notice that here the chaotic region is thin in comparison with the basic quantum phase space cell of area $2\pi\hbar$ and therefore it is not so different from the torus quantized states. Indeed, the wavefunction in configuration space (figure 4(a)) has a quite orderly appearance. Another example of a regular state quantized on a torus around the period-three orbit is shown in figures 4(c) and 4(d), where again the quantum localization regions nicely follow the classical tori, as do the periodic orbits selected by the coincidence criterion.

We did not find any states which would live in the classically regular and irregular regions simultaneously, so the classification of eigenstates in the regular and the irregular classes is well founded, supporting the original ideas by Percival (1973) and in agreement

Figure 1. (Opposite) Two eigenstates for $\lambda = 0.375$: the 2002nd even-parity state at energy $E = 12\,591.36$ (a), (b), and the 2010th even-parity state at energy $E = 12\,634.76$ (c), (d). We plot the wavefunctions in configuration space (a) and (c) showing the isodensity contours (upper half) and the nodal lines (lower half). In (b) and (d) we show the constant level contours of the corresponding smoothed Wigner function ρ_{SOS}. The contours go in equal steps starting at 1/8 of the maximal value. The abscissa is just the coordinate on the line of section whilst the ordinate covers the classically allowed region in momentum space, running from $-\hbar\sqrt{E}$ to $\hbar\sqrt{E}$. In (b) and (d) we also plot the classical periodic orbits (little squares) with periods up to $n_{\max} = 20$ which satisfy the coincidence criterion (see text). (The complete list includes 3361 periodic orbits.) In (e) we show the Wigner function (d) with reduced smoothing by a factor 2.5 and we also show the classical stable and unstable manifolds of the period-two orbit. Here the negative value regions become readily apparent and their contours are plotted with thin lines.


(a)
(b)
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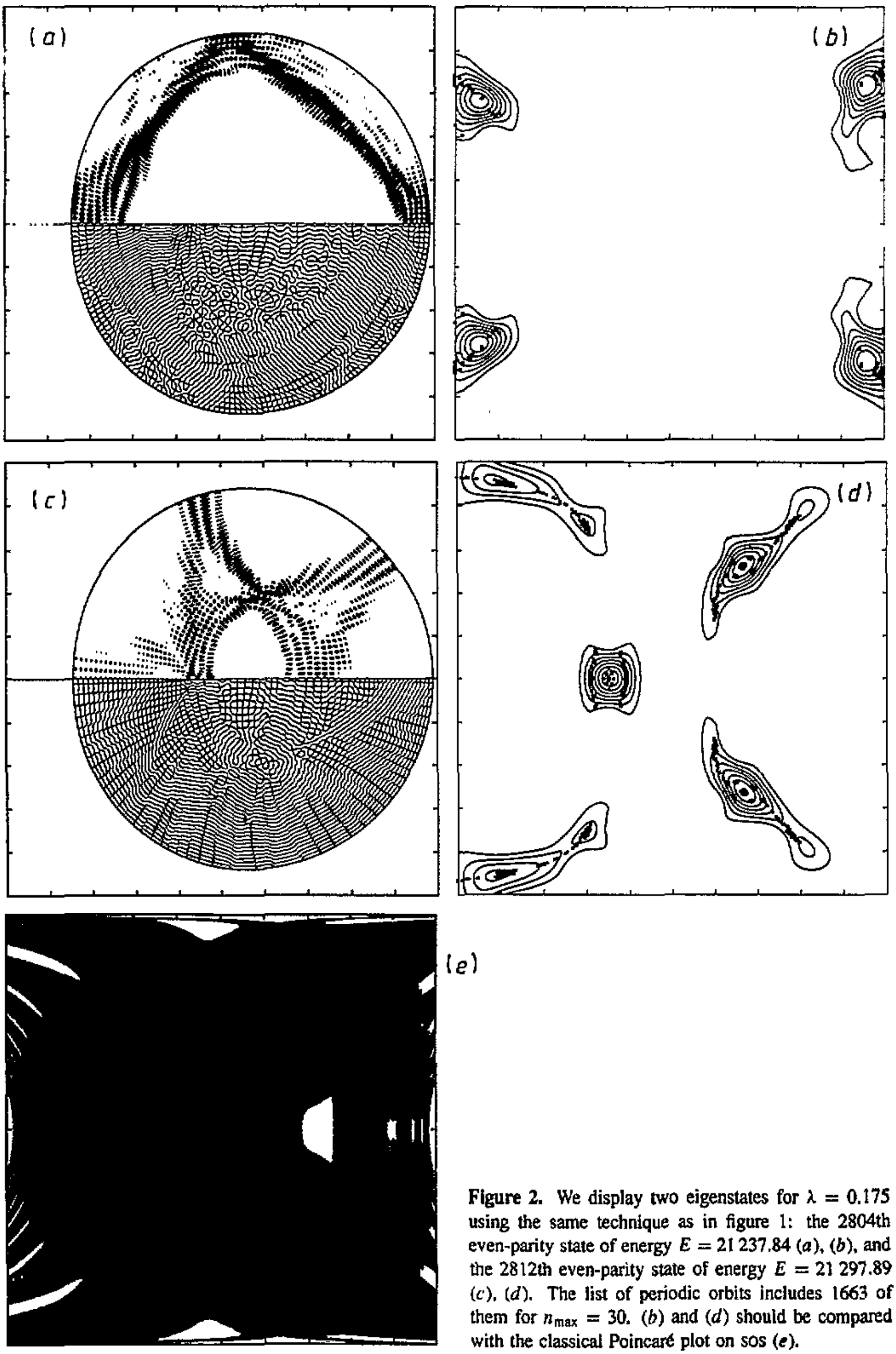


Figure 2. We display two eigenstates for $\lambda = 0.175$ using the same technique as in figure 1: the 2804th even-parity state of energy $E = 21\,237.84$ (*a*), (*b*), and the 2812th even-parity state of energy $E = 21\,297.89$ (*c*), (*d*). The list of periodic orbits includes 1663 of them for $n_{max} = 30$. (*b*) and (*d*) should be compared with the classical Poincaré plot on sos (*e*).

with the recent findings of Bohigas *et al* (1990a,b). The existence of such a clear cut classification of eigenstates justifies the assumptions implicit in the derivation of the semiclassical Berry–Robnik (1984) formulae for the level spacing distribution (cf Prosen and Robnik 1993a).

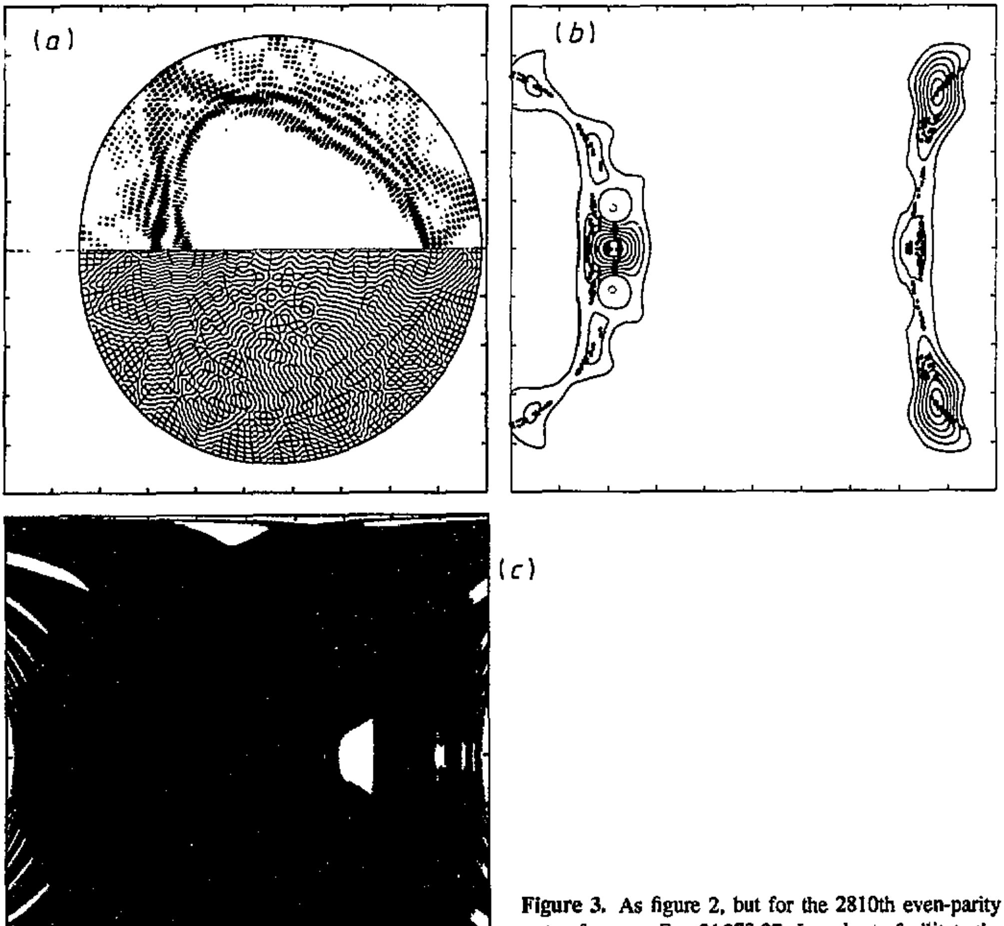
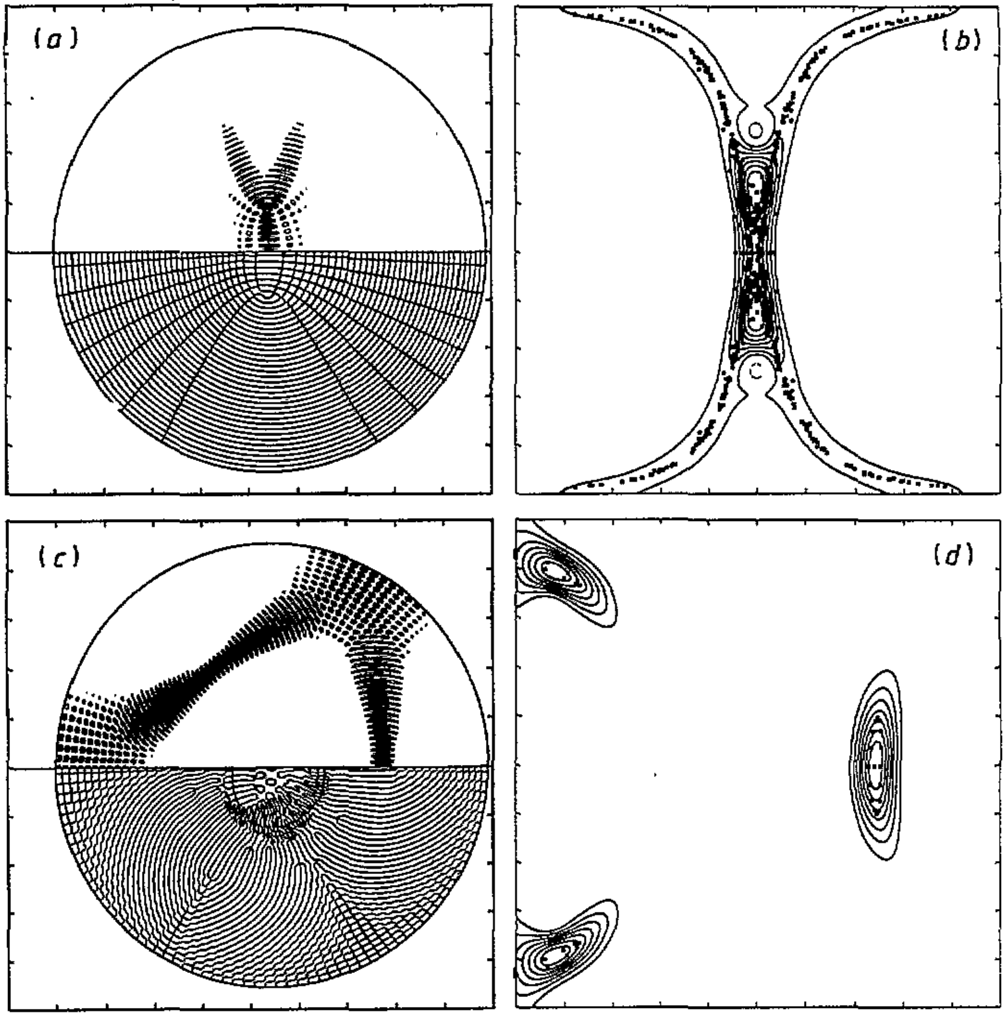


Figure 3. As figure 2, but for the 2810th even-parity state of energy $E = 21\,273.27$. In order to facilitate the comparison of (b) with the classical dynamics we show again classical sos in (c).

Finally, we should mention results on the *global* probability amplitude distribution in the fully chaotic (almost ergodic) regime for $\lambda = 0.375$. Similarly to Aurich and Steiner (1991, 1993), we confirm that almost all states are very well described by the Gaussian distribution, however, the deviations are manifested mainly in significant positive values of the kurtosis $K = \langle(\psi - \langle\psi\rangle)^4\rangle/\langle\psi^2\rangle^2 - 3$; for example for the scar shown in figures 1(c) and 1(d) we obtain $K = 0.84$, but for most of the other states K is close to zero as it should be for the Gaussian distribution.

3. Discussion and conclusions

The main conclusion of this paper is the existence of the well founded classification scheme of eigenstates into regular and irregular states. As for the chaotic wavefunctions we see even



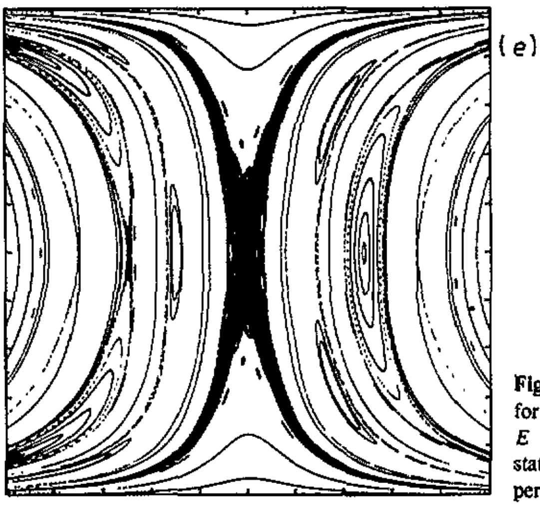


Figure 4. As figure 2, but showing two eigenstates for $\lambda = 0.1$: the 2002nd even-parity state of energy $E = 15\,797.68$ (*a*), (*b*), and the 2018th even-parity state of energy $E = 15\,916.77$ (*c*), (*d*). The list of periodic orbits includes 524 of them for $n_{\max} = 40$.

in high-lying states notable deviations from the microcanonical distribution, but nevertheless very rarely find intensely scarred states. Moreover, the scars can be roughly classified as belonging to one of the three phenomenological classes, as explained above in detail, according to which there exists a clear association of scars with the classical periodic orbits. At the beginning of our work this phenomenological evidence led us to speculate that we might be able to establish a semiclassical theory of *individual eigenstates*. In the meantime, it became obvious that it is impossible even to predict the individual eigenenergies within a vanishing fraction of the mean level spacing by using the semiclassical methods (Prosen and Robnik 1993b). Our conclusion is that the structure of individual eigenstates cannot be predicted by a semiclassical theory, but which is nevertheless useful in describing the collective and the statistical properties of states. We believe that an *a posteriori* modelling of eigenstates in terms of classical periodic orbits in the spirit of the Gutzwiller theory would be useful, but it would necessarily have to include large families of orbits rather than taking into account only one periodic orbit.

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